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2002 J. Phys. A: Math. Gen. 35 1095

(http://iopscience.iop.org/0305-4470/35/4/320)

INSTITUTE OF PHYSICS PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND GENERAL

J. Phys. A: Math. Gen. **35** (2002) 1095–1108 PII: S0305-4470(02)24030-5

Bäcklund transformations for high-order constrained flows of the AKNS hierarchy: canonicity and spectrality property

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Received 16 April 2001, in final form 20 November 2001
Published 18 January 2002
Online at stacks.iop.org/JPhysA/35/1095

Abstract
A new infinite number of one- and two-point Bäcklund transformations (BTs) with explicit expressions are constructed for the high-order constrained flows of the AKNS hierarchy. It is shown that these BTs are canonical transformations including the Bäcklund parameter η and a spectrality property holds with respect to η and the 'conjugated' variable μ for which the point (η, μ) belongs to the spectral curve. In addition, the formulae of m-times repeated Darboux transformations for the high-order constrained flows of the AKNS hierarchy are presented.

PACS numbers: 02.30.Uu, 02.30.Zz

1. Introduction

Bäcklund transformations (BTs) are an important aspect of the theory of integrable systems [1, 2]. It is well known that the BTs for soliton equations are canonical transformations (see, for example, [3–5]). More recently there has been much interest in the properties of BTs for finite-dimensional integrable Hamiltonian systems [6–10]. These BTs are defined as symplectic integrable maps which can be described explicitly and can be viewed as a time discretization of particular flows of Liouville integrable systems. They are canonical transformations including the Bäcklund parameter η. For these BTs a spectrality property holds with respect to η and the 'conjugated' variable μ and the points (η, μ) or $(\eta, f(\mu))$ for some function $f(\mu)$ lie on the spectral curve. An important application of the spectrality property of BTs is that to the problem of separation of variables. In fact, the sequence of Bäcklund parameters η_j together with the conjugate variables μ_j constitute the separation variables for the finite-dimensional integrable Hamiltonian systems [6].

We proceed to develop the ideas with some new BTs and study the problem of constructing one- and two-point BTs for the high-order constrained flows of the soliton hierarchy [11–13]. The Lax representation for the high-order constrained flows can always be deduced from the adjoint representation for the soliton hierarchy [14, 15]. Then, based on the results of Darboux transformations (DTs) for the soliton hierarchy [2, 16–18, 20], we can find the DTs for the high-order constrained flows. By using the Lax representation, these DTs give rise to explicit one- and two-point BTs including one and two Bäcklund parameters η_i, respectively. We show these BTs to be canonical transformations by presenting their generating functions. Then we show that these BTs possess the spectrality property with respect to η and conjugate variable μ, and the pairs (η_i, μ_i) belong to the spectral curve, namely they satisfy the separation equations. A few examples of this kind of BT were presented in [6–10]. This paper presents a way of finding an infinite number of BTs with the property described in [6–10] by means of DTs for the high-order constrained flows of soliton hierarchy. We will use the high-order constrained flows of the AKNS hierarchy to illustrate the ideas.

In section 2, we briefly describe the high-order constrained flows of the AKNS hierarchy. In section 3 we first present three kinds of DTs for the constrained flows of the AKNS hierarchy. Then we find an infinite number of new one-point and two-point BTs from the first and third kind of DTs, respectively, and show them to be canonical transformations possessing the spectrality property by first using the three high-order constrained flows as modelled in sections 3 and 4, respectively. Finally, the formulae for m-times repeated DTs for the constrained flows are presented in section 5.

2. High-order constrained flows of the AKNS hierarchy

Let us briefly describe the high-order constrained flows of AKNS hierarchy. Consider the AKNS spectral problem [21]

$$\psi_x = U(u,\lambda)\psi \equiv \begin{pmatrix} -\lambda & q \\ r & \lambda \end{pmatrix}\psi \qquad \psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \qquad u = \begin{pmatrix} q \\ r \end{pmatrix} \tag{2.1}$$

and the evolution of ψ

$$\psi_{t_n} = V^{(n)}(u,\lambda)\psi = \sum_{i=0}^{n} \begin{pmatrix} a_i & b_i \\ c_i & -a_i \end{pmatrix} \lambda^{n-i}\psi \tag{2.2}$$

where

$$a_0 = -1 \qquad b_0 = c_0 = a_1 = 0 \qquad b_1 = q \qquad c_1 = r$$
$$a_2 = \tfrac{1}{2}qr \qquad b_2 = -\tfrac{1}{2}q_x \qquad c_2 = \tfrac{1}{2}r_x \qquad \ldots$$

and in general

$$\begin{pmatrix} c_{m+1} \\ b_{m+1} \end{pmatrix} = L\begin{pmatrix} c_m \\ b_m \end{pmatrix} = L^m\begin{pmatrix} r \\ q \end{pmatrix} \qquad a_{m,x} = qc_m - rb_m$$

$$L = \frac{1}{2}\begin{pmatrix} D - 2rD^{-1}q & 2rD^{-1}r \\ -2qD^{-1}q & -D + 2qD^{-1}r \end{pmatrix} \qquad D = \frac{\partial}{\partial x} \qquad DD^{-1} = D^{-1}D = 1.$$

Then the compatibility condition of equations (2.1) and (2.2) gives rise to the AKNS hierarchy [21]

$$u_{t_n} = \begin{pmatrix} q \\ r \end{pmatrix}_{t_n} = J\begin{pmatrix} c_{n+1} \\ b_{n+1} \end{pmatrix} = J\frac{\delta H_{n+1}}{\delta u} \qquad n = 0, 1, \ldots \tag{2.3}$$

where

$$H_n = \frac{2a_{n+1}}{n} \qquad J = \begin{pmatrix} 0 & -2 \\ 2 & 0 \end{pmatrix}.$$

We have

$$\frac{\delta\lambda}{\delta q} = \psi_2^2 \qquad \frac{\delta\lambda}{\delta r} = -\psi_1^2.$$

The high-order constrained flows of the AKNS hierarchy consist of the equations obtained from the spectral problem (2.1) for N distinct λ_j and the restriction of the variational derivatives for conserved quantities H_n and λ_j [11, 15],

$$\Phi_{1,x} = -\Lambda\Phi_1 + q\Phi_2 \qquad \Phi_{2,x} = r\Phi + \Lambda\Phi_2 \tag{2.4a}$$

$$\frac{\delta H_{n+1}}{\delta u} - \alpha\sum_{j=1}^{N}\frac{\delta\lambda_j}{\delta u} = \begin{pmatrix} c_{n+1} \\ b_{n+1} \end{pmatrix} - \alpha\begin{pmatrix} \langle\Phi_2, \Phi_2\rangle \\ -\langle\Phi_1, \Phi_1\rangle \end{pmatrix} = 0 \tag{2.4b}$$

where we have used $(\phi_{1j}, \phi_{2j})^T$ to denote the solution of (2.1) with $\lambda = \lambda_j$, $j = 1, \ldots, N$, and $\Phi_i = (\phi_{i1}, \ldots, \phi_{iN})^T$, $i = 1, 2$, $\Lambda = \mathrm{diag}(\lambda_1, \ldots, \lambda_N)$, and $\langle\cdot, \cdot\rangle$ denotes the inner product. The Lax representation for the constrained flow (2.4) is given by [14, 15]

$$M_x^{(n)} = \left[U, M^{(n)}\right] \tag{2.5}$$

with Lax matrix $M^{(n)}$

$$M^{(n)}(u, \Phi_1, \Phi_2, \lambda) = \begin{pmatrix} A^{(n)} & B^{(n)} \\ C^{(n)} & -A^{(n)} \end{pmatrix} = V^{(n)} + M_0 \tag{2.6}$$

$$M_0 = \alpha\sum_{j=1}^{N}\frac{1}{\lambda - \lambda_j}\begin{pmatrix} \phi_{1j}\phi_{2j} & -\phi_{1j}^2 \\ \phi_{2j}^2 & -\phi_{1j}\phi_{2j} \end{pmatrix}$$

and the Lax pair for (2.4)

$$\psi_x = U(u, \lambda)\psi \tag{2.7}$$

$$M^{(n)}(u, \Phi_1, \Phi_2, \lambda)\psi = \mu\psi. \tag{2.8}$$

The spectral curve Γ,

$$\Gamma:\ \det\left(M^{(n)}(u, \Phi_1, \Phi_2, \lambda) - \mu\right) = 0$$

is

$$\mu^2 = \left(A^{(n)}\right)^2(\lambda) + B^{(n)}(\lambda)C^{(n)}(\lambda). \tag{2.9}$$

We present the first three high-order constrained flows as follows:

(1) For $n = 0$, $\alpha = \frac{1}{2}$, (2.4b) gives an explicit constraint

$$q = -\tfrac{1}{2}\langle\Phi_1, \Phi_1\rangle \qquad r = \tfrac{1}{2}\langle\Phi_2, \Phi_2\rangle. \tag{2.10}$$

Then (2.4a) becomes a finite-dimensional integrable Hamiltonian system (FDIHS):

$$\begin{aligned} &\Phi_{1,x} = \frac{\partial\tilde{H}_0}{\partial\Phi_2} \qquad \Phi_{2,x} = -\frac{\partial\tilde{H}_0}{\partial\Phi_1} \\ &\tilde{H}_0 = -\langle\Lambda\Phi_1, \Phi_2\rangle - \tfrac{1}{4}\langle\Phi_1, \Phi_1\rangle\langle\Phi_2, \Phi_2\rangle \end{aligned} \tag{2.11}$$

with Lax matrix $M^{(0)}$

$$A^{(0)} = -1 + \frac{1}{2}\sum_{j=1}^{N}\frac{1}{\lambda - \lambda_j}\phi_{1j}\phi_{2j} \qquad B^{(0)} = -\frac{1}{2}\sum_{j=1}^{N}\frac{1}{\lambda - \lambda_j}\phi_{1j}^2$$

$$C^{(0)} = \frac{1}{2}\sum_{j=1}^{N}\frac{1}{\lambda - \lambda_j}\phi_{2j}^2. \tag{2.12}$$

The spectral curve Γ is a hyper-elliptic, genus $N-1$ curve

$$\mu^2 = 1 + \sum_{j=1}^{N}\frac{P_j}{\lambda - \lambda_j} \tag{2.13}$$

with

$$P_j = -\phi_{1j}\phi_{2j} + \frac{1}{2}\sum_{k\neq j}\frac{1}{\lambda_j - \lambda_k}\left(\phi_{1j}\phi_{2j}\phi_{1k}\phi_{2k} - \phi_{1k}^2\phi_{2j}^2\right) \qquad j = 1, \ldots, N.$$

$P_1, \ldots, P_N$ are N independent integrals of motion in involution for FDIHS (2.11).

(2) For $n = 1, \alpha = -\frac{1}{4}$, (2.4) can be written as a FDIHS

$$Q_x = \frac{\partial \tilde{H}_1}{\partial P} \qquad P_x = -\frac{\partial \tilde{H}_1}{\partial Q} \tag{2.14}$$

with

$$\tilde{H}_1 = -\langle \Lambda\Phi_1, \Phi_2\rangle - \tfrac{1}{2}r\langle\Phi_1, \Phi_1\rangle + \tfrac{1}{2}q\langle\Phi_2, \Phi_2\rangle$$
$$Q = (\phi_{11}, \ldots, \phi_{1N}, q)^T \qquad P = (\phi_{21}, \ldots, \phi_{2N}, r)^T$$

and Lax matrix $M^{(1)}$

$$A^{(1)} = -\lambda - \frac{1}{4}\sum_{j=1}^{N}\frac{1}{\lambda - \lambda_j}\phi_{1j}\phi_{2j} \qquad B^{(1)} = q + \frac{1}{4}\sum_{j=1}^{N}\frac{1}{\lambda - \lambda_j}\phi_{1j}^2$$
$$C^{(1)} = r - \frac{1}{4}\sum_{j=1}^{N}\frac{1}{\lambda - \lambda_j}\phi_{2j}^2. \tag{2.15}$$

The spectral curve Γ is

$$\mu^2 = \lambda^2 + P_0 + \sum_{j=1}^{N}\frac{P_j}{\lambda - \lambda_j} \tag{2.16}$$

with

$$P_0 = \tfrac{1}{2}\langle\Phi_1\Phi_2\rangle + qr$$
$$P_j = \frac{1}{4}\left(2\lambda_j\phi_{1j}\phi_{2j} + r\phi_{1j}^2 - q\phi_{2j}^2\right) + \frac{1}{8}\sum_{k\neq j}\frac{1}{\lambda_j - \lambda_k}\left(\phi_{1j}\phi_{2j}\phi_{1k}\phi_{2k} - \phi_{1k}^2\phi_{2j}^2\right).$$

$P_0, \ldots, P_N$ are $N+1$ independent integrals of motion in involution for FDIHS (2.14).

(3) For $n = 2, \alpha = \frac{1}{2}$, by introducing the following Jacobi–Ostrogradsky coordinates:

$$Q = (\phi_{11}, \ldots, \phi_{1N}, q_1, q_2)^T \qquad P = (\phi_{21}, \ldots, \phi_{2N}, r_1, r_2)^T$$
$$q_1 = q \qquad q_2 = r \qquad p_1 = -\tfrac{1}{4}r_x \qquad p_2 = -\tfrac{1}{4}q_x$$

equation (2.4) can be transformed into a FDIHS

$$Q_x = \frac{\partial \tilde{H}_2}{\partial P} \qquad P_x = -\frac{\partial \tilde{H}_2}{\partial Q} \tag{2.17}$$

with

$$\tilde{H}_2 = -\langle \Lambda\Phi_1, \Phi_2\rangle - \tfrac{1}{2}q_2\langle \Phi_1, \Phi_1\rangle + \tfrac{1}{2}q_1\langle \Phi_2, \Phi_2\rangle + \tfrac{1}{4}q_1^2q_2^2 - 4p_1p_2$$

and Lax matrix $M^{(2)}$

$$A^{(2)} = -\lambda^2 + \frac{1}{2}q_1q_2 + \frac{1}{2}\sum_{j=1}^{N}\frac{1}{\lambda-\lambda_j}\phi_{1j}\phi_{2j} \qquad B^{(2)} = q_1\lambda + 2p_2 - \frac{1}{2}\sum_{j=1}^{N}\frac{1}{\lambda-\lambda_j}\phi_{1j}^2$$
$$C^{(2)} = q_2\lambda - 2p_1 + \frac{1}{2}\sum_{j=1}^{N}\frac{1}{\lambda-\lambda_j}\phi_{2j}^2. \tag{2.18}$$

The spectral curve Γ is

$$\mu^2 = \lambda^4 + P_0\lambda + P_{N+1} + \sum_{j=1}^{N}\frac{P_j}{\lambda-\lambda_j} \tag{2.19}$$

with

$$P_0 = -\langle\Phi_1\Phi_2\rangle - 2q_1p_1 + 2p_2q_2 \qquad P_{N+1} = \tilde{H}_2$$
$$P_j = \frac{1}{2}\left(-2\lambda_j^2\phi_{1j}\phi_{2j} - \lambda_jq_2\phi_{1j}^2 + \lambda_jq_1\phi_{2j}^2 + q_1q_2\phi_{1j}\phi_{2j}\right) + p_1\phi_{1j}^2 + p_2\phi_{2j}^2$$
$$+\frac{1}{2}\sum_{k\neq j}\frac{1}{\lambda_j-\lambda_k}\left(\phi_{1j}\phi_{2j}\phi_{1k}\phi_{2k} - \phi_{1k}^2\phi_{2j}^2\right) \qquad j = 1, \ldots, N.$$

$P_0, \ldots, P_{N+1}$ are $N+2$ independent integrals of motion in involution for FDIHS (2.17).

3. One-point BTs for high-order constrained flows of the AKNS hierarchy

We first briefly review DTs for the AKNS hierarchy. Suppose that a gauge transformation

$$\bar{\psi} = T\psi \tag{3.1}$$

transforms (2.1) and (2.2) into

$$\bar{\psi}_x = \bar{U}(\bar{u}, \lambda)\bar{\psi} \tag{3.2}$$
$$\bar{\psi}_{t_n} = \bar{V}^{(n)}(\bar{u}, \lambda)\bar{\psi}. \tag{3.3}$$

Let $\psi(x, \eta_i)$ be a solution of (2.1) and (2.2) with $\lambda = \eta_i, i = 1, 2, \eta_i \neq \lambda_j$. It is known [2, 16–18] that there are the following three kinds of DTs for the AKNS hierarchy.

(1) The first DT for the AKNS hierarchy is given by

$$T_1 = \begin{pmatrix} \lambda - \eta_1 + \frac{1}{2}qf_1 & -\frac{1}{2}q \\ -f_1 & 1 \end{pmatrix} \qquad f_i = \frac{\psi_2(x, \eta_i)}{\psi_1(x, \eta_i)} \tag{3.4}$$

and

$$\bar{q} = -\tfrac{1}{2}q_x - \eta_1 q + \tfrac{1}{2}q^2 f_1 \qquad \bar{r} = 2f_1 \tag{3.5}$$

namely under the transformation (3.1) with (3.4) and (3.5), $\bar{U}$ and $\bar{V}^{(n)}$ are of the same form as U and $V^{(n)}$ except for replacing q and r by $\bar{q}$ and $\bar{r}$. So (3.5) presents the relationship between two solutions (q, r) and $(\bar{q}, \bar{r})$ of equation (2.3).

(2) The second DT for the AKNS hierarchy is given by

$$T_2 = \begin{pmatrix} 1 & -f_2 \\ \frac{1}{2}r & \lambda - \eta_2 - \frac{1}{2}rf_2 \end{pmatrix} \tag{3.6}$$

and

$$\bar{q} = -2f_2 \qquad \bar{r} = \tfrac{1}{2}r_x - \eta_2 r - \tfrac{1}{2}r^2 f_2. \tag{3.7}$$

(3) The third DT for the AKNS hierarchy is given by

$$T_3 = \begin{pmatrix} \lambda - \eta_1 + m_2 & -m_1 \\ m_3 & \lambda - \eta_2 - m_2 \end{pmatrix} \tag{3.8}$$

and

$$\bar{q} = q - 2m_1, \qquad \bar{r} = r - 2m_3 \tag{3.9}$$

with

$$\begin{aligned} m_1 &= \frac{(\eta_2 - \eta_1)\psi_1(\eta_1)\psi_1(\eta_2)}{\triangle} \qquad & m_2 &= \frac{(\eta_2 - \eta_1)\psi_1(\eta_2)\psi_2(\eta_1)}{\triangle} \\ m_3 &= \frac{(\eta_2 - \eta_1)\psi_2(\eta_1)\psi_2(\eta_2)}{\triangle} \qquad & \triangle &= \psi_1(\eta_1)\psi_2(\eta_2) - \psi_2(\eta_1)\psi_1(\eta_2). \end{aligned} \tag{3.10}$$

We now consider the DTs for high-order constrained flows (2.4). Assuming the gauge transformation (3.1) then accordingly

$$\begin{pmatrix} \bar{\phi}_{1j} \\ \bar{\phi}_{2j} \end{pmatrix} = \beta_j T \begin{pmatrix} \phi_{1j} \\ \phi_{2j} \end{pmatrix} \tag{3.11}$$

transforms (2.7) and (2.8) into

$$\bar{\psi}_x = \bar{U}(\bar{u}, \lambda)\bar{\psi} \tag{3.12}$$

$$\bar{M}^{(n)}(\bar{u}, \bar{\Phi}_1, \bar{\Phi}_2, \lambda)\bar{\psi} = \mu\bar{\psi} \tag{3.13}$$

where $\bar{U}$ and $\bar{M}^{(n)}$ satisfy

$$T_x = \bar{U}(\bar{u}, \lambda)T - TU(u, \lambda) \tag{3.14}$$

$$\bar{M}^{(n)}(\bar{u}, \bar{\Phi}_1, \bar{\Phi}_2, \lambda)T = TM^{(n)}(u, \Phi_1, \Phi_2, \lambda). \tag{3.15}$$

Motivated by the first DT for the AKNS hierarchy, let $\psi(x, \eta_i)$ be a solution of (2.7) and (2.8) with $\lambda = \eta_i,\ \mu = \mu_i,\ i = 1, 2,\ \eta_i \neq \lambda_j$. We find that the first DT for the constrained flows (2.4) consists of (3.1), (3.4), (3.5) and (3.11) with $\beta_j = \frac{1}{\sqrt{\lambda_j - \eta_1}}$, namely

$$\bar{\phi}_{1j} = \sqrt{\lambda_j - \eta_1}\phi_{1j} - \frac{1}{2\sqrt{\lambda_j - \eta_1}}q(\phi_{2j} - f_1\phi_{1j}) \tag{3.16a}$$

$$\bar{\phi}_{2j} = \frac{1}{\sqrt{\lambda_j - \eta_1}}(\phi_{2j} - f_1\phi_{1j}) \qquad j = 1, \ldots, N. \tag{3.16b}$$

In fact, based on the results of the DTs for the AKNS hierarchy, it can be shown by using a similar method as presented in [19, 20] that under the transformations (3.1), (3.4), (3.5) and (3.16), $\bar{U}$ and $\bar{M}^{(n)}$ permit the same form as U and $M^{(n)}$ except for replacing $q, r, \phi_{1j}, \phi_{2j}$ by $\bar{q}, \bar{r}, \bar{\phi}_{1j}, \bar{\phi}_{2j}$, namely, we have

$$T_x = U(\bar{u}, \lambda)T - TU(u, \lambda) \tag{3.17}$$

$$M^{(n)}(\bar{u}, \bar{\Phi}_1, \bar{\Phi}_2, \lambda)T = TM^{(n)}(u, \Phi_1, \Phi_2, \lambda). \tag{3.18}$$

The equalities (3.17) and (3.18) ensure that (2.7) and (2.8) are invariant under the transformations (3.1), (3.4), (3.5) and (3.16). This guarantees that the relationship between $q, r, \phi_{1j}, \phi_{2j}$ and $\bar{q}, \bar{r}, \bar{\phi}_{1j}, \bar{\phi}_{2j}$ obtained from (3.17) and (3.18) is just the one between two solutions of the constrained flows (2.4). In fact, (3.17) and (3.18) give rise to (3.5) and (3.16) which present the relationship between two solutions of the constrained flows (2.4).

It follows from (2.8)

$$f_i = \frac{\mu_i - A^{(n)}(\eta_i)}{B^{(n)}(\eta_i)} = \frac{C^{(n)}(\eta_i)}{\mu_i + A^{(n)}(\eta_i)} \qquad i = 1, 2. \tag{3.19}$$

By substituting (3.19) into (3.5) and (3.16), we obtain an infinite number ($n = 0, 1, \ldots$) of the first explicit one-point BT B_{η_1} for the constrained flows (2.4) as follows:

$$\bar{q} = -\frac{1}{2}q_x - \eta_1 q + \frac{1}{2}q^2 \frac{\mu_1 - A^{(n)}(\eta_1)}{B^{(n)}(\eta_1)} \qquad \bar{r} = 2\frac{\mu_1 - A^{(n)}(\eta_1)}{B^{(n)}(\eta_1)} \tag{3.20a}$$

$$\bar{\phi}_{1j} = \sqrt{\lambda_j - \eta_1}\phi_{1j} - \frac{q}{2\sqrt{\lambda_j - \eta_1}}\left(\phi_{2j} - \frac{\mu_1 - A^{(n)}(\eta_1)}{B^{(n)}(\eta_1)}\phi_{1j}\right) \tag{3.20b}$$

$$\bar{\phi}_{2j} = \frac{1}{\sqrt{\lambda_j - \eta_1}}\left(\phi_{2j} - \frac{\mu_1 - A^{(n)}(\eta_1)}{B^{(n)}(\eta_1)}\phi_{1j}\right). \tag{3.20c}$$

It is found from (3.4) and (3.18)

$$(\lambda - \eta_1)\bar{A}^{(n)}(\lambda) = (\lambda - \eta_1 + qf_1)A^{(n)}(\lambda) + f_1\left(\lambda - \eta_1 + \tfrac{1}{2}qf_1\right)B^{(n)}(\lambda) - \tfrac{1}{2}qC^{(n)}(\lambda) \tag{3.21a}$$

$$(\lambda - \eta_1)\bar{B}^{(n)}(\lambda) = q\left(\lambda - \eta_1 + \tfrac{1}{2}qf_1\right)A^{(n)}(\lambda) + \left(\lambda - \eta_1 + \tfrac{1}{2}qf_1\right)^2 B^{(n)}(\lambda) - \tfrac{1}{4}q^2C^{(n)}(\lambda) \tag{3.21b}$$

$$(\lambda - \eta_1)\bar{C}^{(n)}(\lambda) = -2f_1A^{(n)}(\lambda) - f_1^2B^{(n)}(\lambda) + C^{(n)}(\lambda) \tag{3.21c}$$

which, as we mentioned above, present the relationship between two solutions of the constrained flows (2.4).

Using the first three constrained flows as a model, we now show the BTs (3.20) to be canonical transformations by presenting their generating functions and check the spectrality property with respect to the Bäcklund parameter η and the 'conjugated' variable μ with the point (η, μ) belonging to the spectral curve (2.9).

(1) For the first constrained flow, the FDIHS (2.11), using (2.12) and comparing the coefficients of λ^0 in (3.21 c), one gets

$$f_1 = \tfrac{1}{4}\langle\bar{\Phi}_2, \bar{\Phi}_2\rangle. \tag{3.22}$$

Then we have from (3.16)

$$\phi_{2j} = \sqrt{\lambda_j - \eta_1}\bar{\phi}_{2j} + \frac{1}{4}\langle\bar{\Phi}_2, \bar{\Phi}_2\rangle\phi_{1j} = \frac{\partial S^{(0)}}{\partial\phi_{1j}} \tag{3.23a}$$

$$\bar{\phi}_{1j} = \sqrt{\lambda_j - \eta_1}\phi_{1j} + \frac{1}{4}\langle\Phi_1, \Phi_1\rangle\bar{\phi}_{2j} = \frac{\partial S^{(0)}}{\partial\bar{\phi}_{2j}} \tag{3.23b}$$

where the generating function $S^{(0)}$ for the canonical transformation (3.20) is given by

$$S^{(0)} = \frac{1}{8}\langle\Phi_1, \Phi_1\rangle\langle\bar{\Phi}_2, \bar{\Phi}_2\rangle + \sum_{j=1}^{N}\sqrt{\lambda_j - \eta_1}\phi_{1j}\bar{\phi}_{2j} - \eta_1. \tag{3.24}$$

Furthermore, it is found from (2.12) and (3.16)

$$\begin{aligned}\frac{\partial S^{(0)}}{\partial\eta_1} &= -1 - \frac{1}{2}\sum_{j=1}^{N}\frac{1}{\sqrt{\lambda_j - \eta_1}}\phi_{1j}\bar{\phi}_{2j} \\ &= -1 - \frac{1}{2}\sum_{j=1}^{N}\frac{1}{\sqrt{\lambda_j - \eta_1}}\phi_{1j}\frac{1}{\sqrt{\lambda_j - \eta_1}}[\phi_{2j} - f_1\phi_{1j}] \\ &= A^{(0)}(\eta_1) + f_1B^{(0)}(\eta_1) = \mu_1\end{aligned} \tag{3.25}$$

which implies that (η_1, μ_1) satisfies the spectrality property. Consider the composition $B_{\eta_1\ldots\eta_N} = B_{\eta_1} \circ \cdots \circ B_{\eta_N}$ of the BT B_{η_i}. Then the corresponding generating function $S^{(0)}_{\eta_1\ldots\eta_N}$ becomes the generating function of the canonical transformation from (Φ_1, Φ_2) to (η, μ) given by the equations

$$\phi_{2j} = \frac{\partial S^{(0)}_{\eta_1\ldots\eta_N}}{\partial \phi_{1j}} \qquad \mu_j = \frac{\partial S^{(0)}_{\eta_1\ldots\eta_N}}{\partial \eta_j}.$$

The points (η_i, μ_i) satisfy the separation equations given by the spectral curve (2.13)

$$\mu_i^2 = 1 + \sum_{j=1}^{N} \frac{P_j}{\eta_i - \lambda_j} \qquad i = 1, \ldots, N.$$

(2) For the second constrained flow, the FDIHS (2.14), using (2.15) and comparing the coefficients of λ, λ^0 in (3.21c) and the coefficient of λ in (3.21b), one gets $f_1 = \frac{1}{2}\bar{r}$ and

$$r = -\frac{1}{4}\langle\bar{\Phi}_2, \bar{\Phi}_2\rangle - \eta_1\bar{r} + \frac{1}{4}q\bar{r}^2 = \frac{\partial S^{(1)}}{\partial q} \tag{3.26}$$

$$\bar{q} = \frac{1}{4}\langle\Phi_1, \Phi_1\rangle - \eta_1 q + \frac{1}{4}q^2\bar{r} = \frac{\partial S^{(1)}}{\partial \bar{r}} \tag{3.27}$$

then using (3.16)

$$\phi_{2j} = \sqrt{\lambda_j - \eta_1}\bar{\phi}_{2j} + \frac{1}{2}\bar{r}\phi_{1j} = \frac{\partial S^{(1)}}{\partial \phi_{1j}} \tag{3.28a}$$

$$\bar{\phi}_{1j} = \sqrt{\lambda_j - \eta_1}\phi_{1j} - \frac{1}{2}q\bar{\phi}_{2j} = \frac{\partial S^{(1)}}{\partial \bar{\phi}_{2j}} \tag{3.28b}$$

where the generating function $S^{(1)}$ for the canonical transformation (3.20) is given by

$$S^{(1)} = \frac{1}{4}\bar{r}\langle\Phi_1, \Phi_1\rangle - \frac{1}{4}q\langle\bar{\Phi}_2, \bar{\Phi}_2\rangle - \eta_1 q\bar{r} + \frac{1}{8}q^2\bar{r}^2 + \sum_{j=1}^{N}\sqrt{\lambda_j - \eta_1}\phi_{1j}\bar{\phi}_{2j} + \eta_1^2. \tag{3.29}$$

Furthermore, it is easy to check the spectrality property by using (3.16) and (2.15)

$$\frac{\partial S^{(1)}}{\partial \eta_1} = -\frac{1}{2}\sum_{j=1}^{N}\frac{1}{\sqrt{\lambda_j - \eta_1}}\phi_{1j}\bar{\phi}_{2j} - q\bar{r} + 2\eta_1 = -2\left[A^{(1)}(\eta_1) + f_1 B^{(1)}(\eta_1)\right] = -2\mu_1. \tag{3.30}$$

The point (η_1, μ_1) satisfies the separation equation given by the spectral curve (2.16)

$$\mu_1^2 = \eta_1^2 + P_0 + \sum_{j=1}^{N}\frac{P_j}{\eta_1 - \lambda_j}.$$

(3) For the third constrained flow, the FDIHS (2.17), using (2.18), (3.16) and (3.21) in the same way as for (3.28), one gets $f_1 = \frac{1}{2}\bar{q}_2$ and

$$\phi_{2j} = \sqrt{\lambda_j - \eta_1}\bar{\phi}_{2j} + \frac{1}{2}\bar{q}_2\phi_{1j} = \frac{\partial S^{(2)}}{\partial \phi_{1j}} \tag{3.31a}$$

$$\bar{\phi}_{1j} = \sqrt{\lambda_j - \eta_1}\phi_{1j} - \frac{1}{2}q_1\bar{\phi}_{2j} = \frac{\partial S^{(2)}}{\partial \bar{\phi}_{2j}} \tag{3.31b}$$

$$\bar{q}_1 = \frac{1}{4}q_1^2\bar{q}_2 - \eta_1 q_1 + 2p_2 = \frac{\partial S^{(2)}}{\partial \bar{p}_1} \tag{3.31c}$$

$$q_2 = \frac{1}{4}q_1\bar{q}_2^2 - \eta_1\bar{q}_2 - 2\bar{p}_1 = -\frac{\partial S^{(2)}}{\partial p_2} \tag{3.31d}$$

then $p_1 = -\frac{1}{4}q_{2x}$ and $\bar{p}_2 = -\frac{1}{4}\bar{q}_{1x}$ lead to

$$p_1 = -\frac{1}{4}\langle\bar{\Phi}_2, \bar{\Phi}_2\rangle - \eta_1\bar{p}_1 + \frac{1}{2}q_1\bar{q}_2\bar{p}_1 - \frac{1}{16}q_1^2\bar{q}_2^3 + \frac{1}{4}q_1\bar{q}_2^2\eta_1 - \frac{1}{4}\bar{q}_2^2p_2 = \frac{\partial S^{(2)}}{\partial q_1} \tag{3.31e}$$

$$\bar{p}_2 = -\frac{1}{4}\langle\Phi_1, \Phi_1\rangle - \eta_1 p_2 + \frac{1}{2}q_1\bar{q}_2 p_2 + \frac{1}{16}q_1^3\bar{q}_2^2 - \frac{1}{4}q_1^2\bar{q}_2\eta_1 - \frac{1}{4}q_1^2\bar{p}_1 = -\frac{\partial S^{(2)}}{\partial \bar{q}_2} \tag{3.31f}$$

where the generating function $S^{(2)}$ for the canonical transformation (3.20) is given by

$$S^{(2)} = \frac{1}{4}\bar{q}_2\langle\Phi_1, \Phi_1\rangle - \frac{1}{4}q_1\langle\bar{\Phi}_2, \bar{\Phi}_2\rangle - \eta_1 q_1\bar{p}_1 + \eta_1\bar{q}_2 p_2 - \frac{1}{4}q_1\bar{q}_2^2 p_2 + 2\bar{p}_1 p_2 + \frac{1}{4}q_1^2\bar{q}_2\bar{p}_1$$
$$+ \frac{1}{8}q_1^2\bar{q}_2^2\eta_1 - \frac{1}{48}q^3\bar{q}_2^3 + \sum_{j=1}^{N}\sqrt{\lambda_j - \eta_1}\,\phi_{1j}\bar{\phi}_{2j} - \frac{1}{3}\eta_1^3. \tag{3.32}$$

Then it is easy to check the spectrality

$$\frac{\partial S^{(2)}}{\partial \eta_1} = \frac{1}{2}\sum_{j=1}^{N}\frac{\phi_{1j}}{\lambda_j - \eta_1}\left[\phi_{2j} - \frac{1}{2}\bar{q}_2\phi_{1j}\right] + \bar{q}_2 p_2 + \frac{1}{2}q_1\left[q_2 - \frac{1}{4}q_1\bar{q}_2^2 + \bar{q}_2\eta_1\right] + \frac{1}{8}q_1^2\bar{q}_2^2$$
$$= A^{(2)}(\eta_1) + f_1 B^{(2)}(\eta_1) = \mu_1. \tag{3.33}$$

The point (η_1, μ_1) satisfies the separation equation given by the spectral curve (2.19)

$$\mu_1^2 = \eta_1^4 + P_0\eta_1 + P_{N+1} + \sum_{j=1}^{N}\frac{P_j}{\eta_1 - \lambda_j}.$$

In exactly the same way, we can find the second one-point BTs for the constrained flows (2.4) according to the second DTs for the AKNS hierarchy. By composition of these two BTs, we can find two-point BTs for the constrained flows (2.4). Since these two-point BTs are quite complicated, we will present another two-point BT for the constrained flows in section 4.

4. Two-point BTs for high-order constrained flows of the AKNS hierarchy

Let $\psi(x, \eta_i)$ be a solution of (2.7) and (2.8) with $\lambda = \eta_i$, $\mu = \mu_i$, $i = 1, 2$, $\eta_i \neq \lambda_j$. Motivated by the third DT for the AKNS hierarchy, we obtain the third DT for the constrained flows (2.4) consisting of (3.1), (3.8) and

$$\bar{q} = q - 2m_1 \qquad \bar{r} = r - 2m_3 \tag{4.1a}$$

$$\bar{\phi}_{1j} = \frac{1}{\sqrt{(\lambda_j - \eta_1)(\lambda_j - \eta_2)}}[(\lambda_j - \eta_1 + m_2)\phi_{1j} - m_1\phi_{2j}] \tag{4.1b}$$

$$\bar{\phi}_{2j} = \frac{1}{\sqrt{(\lambda_j - \eta_1)(\lambda_j - \eta_2)}}[m_3\phi_{1j} + (\lambda_j - \eta_2 - m_2)\phi_{2j}]. \tag{4.1c}$$

It follows from (3.10) and (3.19):

$$m_1 = \frac{(\eta_2 - \eta_1)B^{(n)}(\eta_1)B^{(n)}(\eta_2)}{\left(\mu_2 - A^{(n)}(\eta_2)\right)B^{(n)}(\eta_1) - \left(\mu_1 - A^{(n)}(\eta_1)\right)B^{(n)}(\eta_2)} \tag{4.2a}$$

$$m_3 = \frac{(\eta_2 - \eta_1)\left(\mu_1 - A^{(n)}(\eta_1)\right)\left(\mu_2 - A^{(n)}(\eta_2)\right)}{\left(\mu_2 - A^{(n)}(\eta_2)\right)B^{(n)}(\eta_1) - \left(\mu_1 - A^{(n)}(\eta_1)\right)B^{(n)}(\eta_2)} \tag{4.2b}$$

$$m_2 = \frac{(\eta_2 - \eta_1)\left(\mu_1 - A^{(n)}(\eta_1)\right)B^{(n)}(\eta_2)}{\left(\mu_2 - A^{(n)}(\eta_2)\right)B^{(n)}(\eta_1) - \left(\mu_1 - A^{(n)}(\eta_1)\right)B^{(n)}(\eta_2)}. \tag{4.2c}$$

Substituting (4.2), (4.1) gives rise to an infinite number ($n = 0, 1, \ldots$) of explicit two-point BTs for the constrained flows (2.4). We now show that the two-point BTs (4.1) are canonical transformations and possess the spectrality property. It is easy to check that

$$m_2^2 + (\eta_2 - \eta_1)m_2 - m_1 m_3 = 0 \qquad f_1 = \frac{m_2}{m_1} \qquad f_2 = \frac{m_2 + \eta_2 - \eta_1}{m_1}. \tag{4.3}$$

Using (4.3), (4.1b) and (4.1c) can be rewritten as

$$\phi_{2j} = \frac{1}{m_1}[(\lambda_j - \eta_1 + m_2)\phi_{1j} - \sqrt{(\lambda_j - \eta_1)(\lambda_j - \eta_2)}\bar{\phi}_{1j}] \tag{4.4a}$$

$$\bar{\phi}_{2j} = \frac{1}{m_1}[\sqrt{(\lambda_j - \eta_1)(\lambda_j - \eta_2)}\phi_{1j} + (-\lambda_j + \eta_2 + m_2)\bar{\phi}_{1j}]. \tag{4.4b}$$

The formulae (3.8), (3.15) and (4.3) lead to

$$\begin{aligned}\left[\lambda^2 - \lambda(\eta_1 + \eta_2) + \eta_1\eta_2\right]\bar{A}^{(n)}(\lambda) &= \left[\lambda^2 - \lambda(\eta_1 + \eta_2) + \eta_1\eta_2 - 2m_1m_3\right]A^{(n)}(\lambda) \\ &\quad - m_3[\lambda - \eta_1 + m_2]B^{(n)}(\lambda) - m_1[\lambda - \eta_2 - m_2]C^{(n)}(\lambda)\end{aligned} \tag{4.5a}$$

$$\begin{aligned}\left[\lambda^2 - \lambda(\eta_1 + \eta_2) + \eta_1\eta_2\right]\bar{B}^{(n)}(\lambda) &= 2m_1[\lambda - \eta_1 + m_2]A^{(n)}(\lambda) \\ &\quad + [\lambda - \eta_1 + m_2]^2 B^{(n)}(\lambda) - m_1^2 C^{(n)}(\lambda)\end{aligned} \tag{4.5b}$$

$$\begin{aligned}\left[\lambda^2 - \lambda(\eta_1 + \eta_2) + \eta_1\eta_2\right]\bar{C}^{(n)}(\lambda) &= 2m_3[\lambda - \eta_2 - m_2]A^{(n)}(\lambda) \\ &\quad - m_3^2 B^{(n)}(\lambda) + [\lambda - \eta_2 - m_2]^2 C^{(n)}(\lambda).\end{aligned} \tag{4.5c}$$

(1) For the first constrained flow, the FDIHS (2.11), using (4.3) and (4.5), one gets

$$\begin{aligned}m_1 &= \tfrac{1}{4}\langle\bar{\Phi}_1, \bar{\Phi}_1\rangle - \tfrac{1}{4}\langle\Phi_1, \Phi_1\rangle \\ m_2 &= \frac{1}{\langle\bar{\Phi}_1, \bar{\Phi}_1\rangle - \langle\Phi_1, \Phi_1\rangle}[\langle\Lambda\bar{\Phi}_1, \bar{\Phi}_1\rangle + \langle\Lambda\Phi_1, \Phi_1\rangle - \eta_2\langle\bar{\Phi}_1, \bar{\Phi}_1\rangle - \eta_1\langle\Phi_1, \Phi_1\rangle \\ &\quad - 2\sum_{j=1}^{N}\sqrt{(\lambda_j - \eta_1)(\lambda_j - \eta_2)}\phi_{1j}\bar{\phi}_{1j}].\end{aligned} \tag{4.6}$$

Then substituting (4.6) into (4.4) gives rise to

$$\phi_{2j} = \frac{\partial S^{(0)}}{\partial\phi_{1j}} \qquad \bar{\phi}_{2j} = -\frac{\partial S^{(0)}}{\partial\bar{\phi}_{1j}} \qquad j - 1, \ldots, N \tag{4.7}$$

where the generating function $S^{(0)}$ for the canonical transformation (4.1) is given by

$$\begin{aligned}S^{(0)} &= \frac{2}{\langle\bar{\Phi}_1, \bar{\Phi}_1\rangle - \langle\Phi_1, \Phi_1\rangle}[\langle\Lambda\bar{\Phi}_1, \bar{\Phi}_1\rangle + \langle\Lambda\Phi_1, \Phi_1\rangle - \eta_2\langle\bar{\Phi}_1, \bar{\Phi}_1\rangle \\ &\quad - \eta_1\langle\Phi_1, \Phi_1\rangle - 2\sum_{j=1}^{N}\sqrt{(\lambda_j - \eta_1)(\lambda_j - \eta_2)}\phi_{1j}\bar{\phi}_{1j}] - \eta_1 - \eta_2.\end{aligned} \tag{4.8}$$

Furthermore, it is found

$$\begin{aligned}\frac{\partial S^{(0)}}{\partial\eta_1} &= -1 + \frac{2}{\langle\bar{\Phi}_1, \bar{\Phi}_1\rangle - \langle\Phi_1, \Phi_1\rangle}\left[-\langle\Phi_1, \Phi_1\rangle + 2\sum_{j=1}^{N}\sqrt{\frac{(\lambda_j - \eta_2)}{(\lambda_j - \eta_1)}}\phi_{1j}\bar{\phi}_{1j}\right] \\ &= A^{(0)}(\eta_1) + f_1 B^{(0)}(\eta_1) = \mu_1\end{aligned} \tag{4.9a}$$

$$\frac{\partial S^{(0)}}{\partial\eta_2} = A^{(0)}(\eta_2) + f_2 B^{(0)}(\eta_2) = \mu_2. \tag{4.9b}$$

This implies that (η_1, μ_1) and (η_2, μ_2) satisfy the spectrality property and the separation equations given by the spectral curve (2.13)

$$\mu_i^2 = 1 + \sum_{j=1}^{N} \frac{P_j}{\eta_i - \lambda_j} \qquad i = 1, 2.$$

(2) For the second constrained flow, the FDIHS (2.14), formula (4.5) gives rise to

$$m_1 = \tfrac{1}{2}(q - \bar{q}) \qquad m_2 = \frac{1}{q + \bar{q}} \left[\tfrac{1}{4}\langle\bar{\Phi}_1, \bar{\Phi}_1\rangle - \tfrac{1}{4}\langle\Phi_1, \Phi_1\rangle + \eta_1 q - \eta_2\bar{q}\right] \tag{4.10}$$

and

$$r = \frac{4}{(q-\bar{q})^2}\left[-\frac{1}{4}\langle\Lambda\bar{\Phi}_1, \bar{\Phi}_1\rangle - \frac{1}{4}\langle\Lambda\Phi_1, \Phi_1\rangle + \frac{1}{4}(\eta_1 + \eta_2)\langle\bar{\Phi}_1, \bar{\Phi}_1\rangle \right.$$
$$\left. - \eta_1\eta_2\bar{q} + (m_2 - \eta_1)^2 q + \frac{1}{2}\sum_{j=1}^{N}\sqrt{(\lambda_j - \eta_1)(\lambda_j - \eta_2)}\phi_{1j}\bar{\phi}_{1j}\right] \tag{4.11a}$$

$$\bar{r} = -2\frac{m_2}{m_1}(m_2 + \eta_2 - \eta_1) + r. \tag{4.11b}$$

By substitution of (4.10), (4.4) and (4.11) can be rewritten as

$$\phi_{2j} = \frac{\partial S^{(1)}}{\partial \phi_{1j}} \qquad \bar{\phi}_{2j} = -\frac{\partial S^{(1)}}{\partial \bar{\phi}_{1j}} \qquad r = \frac{\partial S^{(1)}}{\partial q} \qquad \bar{r} = -\frac{\partial S^{(1)}}{\partial \bar{q}} \tag{4.12}$$

where the generating function $S^{(1)}$ for the canonical transformation (4.1) is given by

$$S^{(1)} = \frac{1}{q-\bar{q}}\left[\langle\Lambda\bar{\Phi}_1, \bar{\Phi}_1\rangle + \langle\lambda\Phi_1, \Phi_1\rangle - 2\sum_{j=1}^{N}\sqrt{(\lambda_j - \eta_1)(\lambda_j - \eta_2)}\phi_{1j}\bar{\phi}_{1j}\right] + \frac{1}{q^2 - \bar{q}^2}$$
$$\times\left[\frac{1}{4}\langle\bar{\Phi}_1, \bar{\Phi}_1\rangle\langle\Phi_1, \Phi_1\rangle - (\eta_1 + \eta_2)q\langle\bar{\Phi}_1, \bar{\Phi}_1\rangle - (\eta_1 + \eta_2)\bar{q}\langle\Phi_1, \Phi_1\rangle \right.$$
$$\left. - \frac{1}{8}\langle\bar{\Phi}_1, \bar{\Phi}_1\rangle^2 - \frac{1}{8}\langle\Phi_1, \Phi_1\rangle^2 + 4\eta_1\eta_2 q\bar{q} - 2\left(\eta_1^2 + \eta_2^2\right)\bar{q}^2\right] - \eta_1^2 - \eta_2^2. \tag{4.13}$$

It is easy to check the spectrality property

$$\frac{\partial S^{(1)}}{\partial \eta_1} = \frac{1}{q^2 - \bar{q}^2}\left[-q\langle\bar{\Phi}_1, \bar{\Phi}_1\rangle - \bar{q}\langle\Phi_1, \Phi_1\rangle - 4\eta_1\bar{q}^2 + 4\eta_2 q\bar{q}\right] - 2\eta_1 + \frac{1}{q-\bar{q}}$$
$$\times\sum_{j=1}^{N}\sqrt{\frac{\lambda_j - \eta_2}{\lambda_j - \eta_1}}\phi_{1j}\bar{\phi}_{1j} = -2\left[A^{(1)}(\eta_1) + f_1 B^{(1)}(\eta_1)\right] = -2\mu_1 \tag{4.14a}$$

$$\frac{\partial S^{(1)}}{\partial \eta_2} = -2\left[A^{(1)}(\eta_2) + f_2 B^{(1)}(\eta_2)\right] = -2\mu_2. \tag{4.14b}$$

The points (η_i, μ_i) satisfy the separation equations given by the spectral curve (2.16)

$$\mu_i^2 = \eta_i^2 + P_0 + \sum_{j=1}^{N}\frac{P_j}{\eta_i - \lambda_j} \qquad i = 1, 2.$$

(3) For the third constrained flow, the FDIHS (2.17), by means of (4.3) and (4.5), one gets

$$m_1 = \tfrac{1}{2}(q_1 - \bar{q}_1) \qquad m_3 = \tfrac{1}{2}(q_2 - \bar{q}_2)$$
$$m_2 = \tfrac{1}{2}(\eta_1 - \eta_2) + \tfrac{1}{2}\sqrt{(\eta_1 - \eta_2)^2 + (q_1 - \bar{q}_1)(q_2 - \bar{q}_2)} \tag{4.15}$$

and

$$p_1 = \frac{2}{(q_1-\bar{q}_1)^2}\Bigg\{-\frac{1}{2}\langle\Lambda\Phi_1,\Phi_1\rangle - \frac{1}{2}\langle\Lambda\bar{\Phi}_1,\bar{\Phi}_1\rangle + \frac{1}{2}(\eta_1+\eta_2)\langle\bar{\Phi}_1,\bar{\Phi}_1\rangle$$
$$- m_1(m_2-\eta_1)^2 q_1 q_2 + (m_2-\eta_1)^2(m_2 q_1 + m_2\bar{q}_1 + \eta_2\bar{q}_1 - \eta_1 q_1)$$
$$+ \sum_{j=1}^{N}\sqrt{(\lambda_j-\eta_1)(\lambda_j-\eta_2)}\phi_{1j}\bar{\phi}_{1j} + \frac{\eta_1\eta_2-(m_2-\eta_1)^2}{\sqrt{(\eta_1-\eta_2)^2+(q_1-\bar{q}_1)(q_2-\bar{q}_2)}}$$
$$\Bigg[-\frac{1}{2}\langle\bar{\Phi}_1,\bar{\Phi}_1\rangle + \frac{1}{2}\langle\Phi_1,\Phi_1\rangle - \frac{1}{4}q_2\left(q_1^2-\bar{q}_1^2\right) + (m_2-\eta_1)^2(q_1+2\bar{q}_1)$$
$$+2(m_2-\eta_1)(\eta_1+\eta_2)\bar{q}_1 + \eta_1\eta_2\bar{q}_1\Bigg]\Bigg\} \tag{4.16a}$$

$$\bar{p}_1 = p_1 + \tfrac{1}{2}(m_2+\eta_2)(q_2+\bar{q}_2) - \tfrac{1}{2}(\eta_1+\eta_2)\bar{q}_2 \tag{4.16b}$$

$$\bar{p}_2 = \frac{1}{2\sqrt{(\eta_1-\eta_2)^2+(q_1-\bar{q}_1)(q_2-\bar{q}_2)}}\Bigg[-\frac{1}{2}\langle\bar{\Phi}_1,\bar{\Phi}_1\rangle + \frac{1}{2}\langle\Phi_1,\Phi_1\rangle - \frac{1}{4}q_2\left(q_1^2-\bar{q}_1^2\right)$$
$$+(m_2-\eta_1)^2(q_1+2\bar{q}_1) + 2(m_2-\eta_1)(\eta_1+\eta_2)\bar{q}_1 + \eta_1\eta_2\bar{q}_1\Bigg] \tag{4.16c}$$

$$p_2 = \bar{p}_2 + \tfrac{1}{2}(\eta_1+\eta_2)q_1 - \tfrac{1}{2}(m_2+\eta_2)(q_1+\bar{q}_1). \tag{4.16d}$$

By inserting (4.15) into (4.4) and (4.16), a straightforward calculation leads to

$$\phi_{2j} = \frac{\partial S^{(2)}}{\partial\phi_{1j}} \qquad \bar{\phi}_{2j} = -\frac{\partial S^{(2)}}{\partial\bar{\phi}_{1j}} \qquad p_1 = \frac{\partial S^{(2)}}{\partial q_1}$$
$$p_2 = \frac{\partial S^{(2)}}{\partial q_2} \qquad \bar{p}_1 = -\frac{\partial S^{(2)}}{\partial\bar{q}_1} \qquad \bar{p}_2 = -\frac{\partial S^{(2)}}{\partial\bar{q}_2} \tag{4.17}$$

where the generating function $S^{(2)}$ for the canonical transformation (4.1) is given by

$$S^{(2)} = \frac{1}{q_1-\bar{q}_1}\Bigg[\langle\Lambda\bar{\Phi}_1,\bar{\Phi}_1\rangle + \langle\Lambda\Phi_1,\Phi_1\rangle - \frac{1}{2}(\eta_1+\eta_2)(\langle\bar{\Phi}_1,\bar{\Phi}_1\rangle + \langle\Phi_1,\Phi_1\rangle)$$
$$+\frac{2}{3}(m_2-\eta_1)^3(q_1+2\bar{q}_1) + 2(m_2-\eta_1)^2(\eta_1+\eta_2)\bar{q}_1 - (\eta_1+\eta_2)\eta_1\eta_2\bar{q}_1$$
$$+\frac{1}{2}\sqrt{(\eta_1-\eta_2)^2+(q_1-\bar{q}_1)(q_2-\bar{q}_2)}(\langle\Phi_1,\Phi_1\rangle - \langle\bar{\Phi}_1,\bar{\Phi}_1\rangle + 2\eta_1\eta_2\bar{q}_1)$$
$$-2\sum_{j=1}^{N}\sqrt{(\lambda_j-\eta_1)(\lambda_j-\eta_2)}\phi_{1j}\bar{\phi}_{1j}\Bigg] + \frac{1}{4}(\eta_1+\eta_2)(q_1-\bar{q}_1)q_2$$
$$-\frac{1}{4}\sqrt{(\eta_1-\eta_2)^2+(q_1-\bar{q}_1)(q_2-\bar{q}_2)}q_2(q_1+\bar{q}_1) + \frac{1}{3}\left(\eta_1^3+\eta_2^3\right). \tag{4.18}$$

By a straightforward calculation, we can show the spectrality property

$$\frac{\partial S^{(2)}}{\partial\eta_1} = A^{(2)}(\eta_1) + f_1 B^{(2)}(\eta_1) = \mu_1 \qquad \frac{\partial S^{(2)}}{\partial\eta_2} = A^{(2)}(\eta_2) + f_2 B^{(2)}(\eta_2) = \mu_2. \tag{4.19}$$

The points (η_i, μ_i) satisfy the separation equations given by the spectral curve (2.19)

$$\mu_i^2 = \eta_i^4 + P_0\eta_i + P_{N+1} + \sum_{j=1}^{N}\frac{P_j}{\eta_i-\lambda_j} \qquad i = 1, 2.$$

5. *m*-times repeated two-point DTs for high-order constrained flows of the AKNS hierarchy

Assume that $(\psi_1(x,\eta_i),\psi_2(x,\eta_i))^T$, $i=1,\ldots,2m$, are solutions of (2.7) and (2.8) with $\lambda=\eta_i$, $\mu=\mu_i$, $i=1,2,\ldots,2m$, $\eta_i\neq\lambda_j$. We use $q[l], r[l], \phi_{1j}[l], \phi_{2j}[l]$ to denote the action of l-times repeated two-point DTs of (4.1) on the initial solution $q[0], r[0], \phi_{1j}[0], \phi_{2j}[0]$. According to (4.1) we have

$$q[l+1]=q[l]-2m_1[l], \qquad r[l+1]=r[l]-2m_3[l] \tag{5.1a}$$

$$\phi_{1j}[l+1]=\frac{1}{\sqrt{(\lambda_j-\eta_{2l+1})(\lambda_j-\eta_{2l+2})}}[(\lambda_j-\eta_{2l+1}+m_2[l])\phi_{1j}[l]-m_1[l]\phi_{2j}[l]] \tag{5.1b}$$

$$\phi_{2j}[l+1]=\frac{1}{\sqrt{(\lambda_j-\eta_{2l+1})(\lambda_j-\eta_{2l+2})}}[m_3[l]\phi_{1j}[l]+(\lambda_j-\eta_{2l+2}-m_2[l])\phi_{2j}[l]]. \tag{5.1c}$$

We denote

$$G_m=\begin{pmatrix}\eta_1^m\psi_1(\eta_1) & \eta_2^m\psi_1(\eta_2) & \eta_3^m\psi_1(\eta_3) & \ldots & \eta_{2m}^m\psi_1(\eta_{2m})\\ \eta_1^{m-1}\psi_1(\eta_1) & \eta_2^{m-1}\psi_1(\eta_2) & \eta_3^{m-1}\psi_1(\eta_3) & \ldots & \eta_{2m}^{m-1}\psi_1(\eta_{2m})\\ \ldots & \ldots & \ldots & \ldots & \ldots\\ \psi_1(\eta_1) & \psi_1(\eta_2) & \psi_1(\eta_3) & \ldots & \psi_1(\eta_{2m})\\ \eta_1^{m-2}\psi_2(\eta_1) & \eta_2^{m-2}\psi_2(\eta_2) & \eta_3^{m-2}\psi_2(\eta_3) & \ldots & \eta_{2m}^{m-2}\psi_2(\eta_{2m})\\ \ldots & \ldots & \ldots & \ldots & \ldots\\ \psi_2(\eta_1) & \psi_2(\eta_2) & \psi_2(\eta_3) & \ldots & \psi_2(\eta_{2m})\end{pmatrix}$$

$$\triangle_m=\begin{pmatrix}\eta_1^{m-1}\psi_1(\eta_1) & \eta_2^{m-1}\psi_1(\eta_2) & \eta_3^{m-1}\psi_1(\eta_3) & \ldots & \eta_{2m}^{m-1}\psi_1(\eta_{2m})\\ \eta_1^{m-2}\psi_1(\eta_1) & \eta_2^{m-2}\psi_1(\eta_2) & \eta_3^{m-2}\psi_1(\eta_3) & \ldots & \eta_{2m}^{m-2}\psi_1(\eta_{2m})\\ \ldots & \ldots & \ldots & \ldots & \ldots\\ \psi_1(\eta_1) & \psi_1(\eta_2) & \psi_1(\eta_3) & \ldots & \psi_1(\eta_{2m})\\ \eta_1^{m-1}\psi_2(\eta_1) & \eta_2^{m-1}\psi_2(\eta_2) & \eta_3^{m-1}\psi_2(\eta_3) & \ldots & \eta_{2m}^{m-1}\psi_2(\eta_{2m})\\ \ldots & \ldots & \ldots & \ldots & \ldots\\ \psi_2(\eta_1) & \psi_2(\eta_2) & \psi_2(\eta_3) & \ldots & \psi_2(\eta_{2m})\end{pmatrix}$$

$$W_m(j)=\begin{pmatrix}\lambda_j^m\phi_{1j}[0] & \eta_1^m\psi_1(\eta_1) & \eta_2^m\psi_1(\eta_2) & \ldots & \eta_{2m}^m\psi_1(\eta_{2m})\\ \lambda_j^{m-1}\phi_{1j}[0] & \eta_1^{m-1}\psi_1(\eta_1) & \eta_2^{m-1}\psi_1(\eta_2) & \ldots & \eta_{2m}^{m-1}\psi_1(\eta_{2m})\\ \ldots & \ldots & \ldots & \ldots & \ldots\\ \phi_{1j}[0] & \psi_1(\eta_1) & \psi_1(\eta_2) & \ldots & \psi_1(\eta_{2m})\\ \lambda_j^{m-1}\phi_{2j}[0] & \eta_1^{m-1}\psi_2(\eta_1) & \eta_2^{m-1}\psi_2(\eta_2) & \ldots & \eta_{2m}^{m-1}\psi_2(\eta_{2m})\\ \ldots & \ldots & \ldots & \ldots & \ldots\\ \phi_{2j}[0] & \psi_2(\eta_1) & \psi_2(\eta_2) & \ldots & \psi_2(\eta_{2m})\end{pmatrix}.$$

Then m-times repeated DTs for the constrained flows (2.4) are given by

$$q[m]=q[0]-2\frac{G_m}{\triangle_m^*} \qquad r[m]=r[0]-2\frac{G_m^*}{\triangle_m^*} \tag{5.2a}$$

$$\phi_{1j}[m]=\frac{1}{\sqrt{\prod_{i=1}^{2m}(\lambda_j-\eta_i)}}\frac{W_m(j)}{\triangle_m} \qquad \phi_{2j}[m]=\frac{1}{\sqrt{\prod_{i=1}^{2m}(\lambda_j-\eta_i)}}\frac{W_m^*(j)}{\triangle_m^*} \tag{5.2b}$$

where G_m^*, $W_m^*(j)$ and $\triangle_m^*$ are obtained by interchanging $\psi_1(x,\eta_i)$ and $\psi_2(x,\eta_i)$, $i=1,\ldots,2m$, $\phi_{1j}(\lambda_j)$ and $\phi_{2j}(\lambda_j)$ in $G_m, W_m(j)$ and $\triangle_M$, respectively. Formula (5.2*a*) was shown in [2,16], (5.2*b*) can be obtained in the same way by using the formulae for Vandermonde-like determinants G_m^*, $W_m^*(j)$ and $\triangle_m^*$ given in [22].

6. Conclusion

Some methods were presented to construct the BTs with the properties described above in [6–10] for a few examples. In this paper we propose a way to construct an infinite number of explicit one- and two-point BTs for high-order constrained flows of soliton hierarchy by means of the Darboux transformations for the constrained flows by using the high-order constrained flows of the AKNS hierarchy as our model. By constructing the generating functions, it is shown that these BTs are canonical transformations including the Bäcklund parameter η and a spectrality property holds with respect to the Bäcklund parameter η and the conjugate variable μ. The pair (η, μ) lies on the spectral curve and satisfies the separation equation. In addition, we present the formula for m-times repeated Darboux transformations for the high-order constrained flows of the AKNS hierarchy.

The method proposed in this paper can be applied to the high-order binary constrained flows of AKNS hierarchy in [23] to find new explicit BTs with canonicity and spectrality.

Acknowledgments

The work described in this paper was supported by a grant from CityU (project no 7001072) and the Special Funds for Chinese Major Basic Research Project ‘Nonlinear Science’.

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